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## Stringy Fuzziness as the Custodian of Time-Space Noncommutativity

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### Abstract

We study aspects of obtaining field theories with noncommuting time-space coordinates as limits of open-string theories in constant electric-field backgrounds. We find that, within the standard closed-string backgrounds, there is an obstruction to decoupling the time-space noncommutativity scale from that of the string fuzziness scale. We speculate that this censorship may be string-theory's way of protecting the causality and unitarity structure. We study the moduli space of the obstruction in terms of the open- and closed-string backgrounds. Cases of both zero and infinite brane tensions as well as zero string couplings are obtained. A decoupling can be achieved formally by considering complex values of the dilaton and inverting the role of space and time in the light cone. This is reminiscent of a black-hole horizon. We study the corresponding supergravity solution in the large- $N$  limit and find that the geometry has a naked singularity at the physical scale of noncommutativity.

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## 1 Introduction

The list of properties of manifolds which are not ambiguous when studied by point particles, but can be ambiguous when probed by strings, has lengthened considerably over the years. The list includes the geometric data (such as the metric), the topology and the dimensionality of the manifold. There are by now many examples where this occurs for both large and small manifolds. Recently, there has been a new addition to this list of potentially ambiguous properties: theories on noncommutative manifolds [1] were found to be equivalent [2] to other theories on commutative manifolds. These theories had a Dirac–Born–Infeld limit and thus contained many derivatives. When viewed as effective theories for open-string dynamics, it seems that the end-points of the open strings are responsible for such noncommutative properties of the space-time coordinates:

$$[x^\mu, x^\nu] = i \theta^{\mu\nu}. \quad (1.1)$$

The main emphasis was on theories for which only spatial coordinates of the manifold were noncommuting, *i.e.*  $\theta^{0i} = 0$  in (1.1). The framework of string theory in the presence of large magnetic-field backgrounds was very useful for such an analysis [3, 4, 5]. Such a theory seems to contain two explicit scales: the string scale  $\sqrt{\alpha'}$  (which we should also be permitted to hope, will eventually be found to be spontaneously generated) and the noncommuting length scale  $\sqrt{|\theta|}$ .

Seiberg and Witten have found a limit of moduli space in which the two scales can be decoupled [2]. This limit is a field theory decoupled from string oscillators on a noncommutative manifold. It is interesting to dare string theory with an extra challenge, that is to try and obtain out of it a field-theory limit that would realize time-space noncommutativity while decoupled from the string scale. We will find that is not possible, at least within standard sigma-model backgrounds.

It seems clear that the time-space noncommutativity requires a reevaluation of the role of the Hamiltonian, of causality and unitarity. On the other hand, string theory has been known in the past to take care of such issues. For example, the fact that D-branes cannot travel at a speed larger than the speed of light reflects itself, in a T-dual picture [6], in the existence of a bound on the possible strength of an electric-field background [7]. Another recent example is the realization in ref. [8] that standard open strings in flat Minkowski space show the same paradoxical features expected from a time-space noncommutative theory, most notably an apparently acausal behaviour in scattering experiments.

Indeed, there is a time-space uncertainty principle (see [9] for a review and a collection of references), derived on heuristic grounds, which is valid in principle for *both* open- and closed-string theories:

$$|\Delta t \Delta x| \gtrsim \alpha'. \quad (1.2)$$

It operates at the string scale and should apply independently of the background, provided it is sufficiently smooth on the string scale.

On the other hand, the peculiar noncommutativity properties we are interested in are specifically associated to open-string end-points [10]. Therefore, it would be very interesting to disentangle these effects from whatever is masked by ‘standard’ stringy fuzziness in (1.2).

Following the strategy of the purely spatial noncommutative examples, taking the Seiberg–Witten limit of the corresponding noncommutative string backgrounds, one finds an obstruction of the same nature as the maximal electric field. It is not possible to decouple the string fuzziness

from the time-space noncommutativity. It is as if the string fuzziness served as a custodian of causality and unitarity and, as such, could not be decoupled from the noncommutativity scale.

The structure of the paper is as follows: in section 2 we encounter a maximal electric-field obstruction to decoupling the time-space noncommutativity and the string scale. We analyse the moduli space of such backgrounds in terms of both open- and closed-strings geometrical data. We analyse properties of these two families of backgrounds and find that the string theory cannot be decoupled from the field theory. This leads us to suggest in section 3 that time-space noncommutativity is ‘censored’ by the string fuzziness. We further explore this notion in section 4, where we do a formal continuation of the string data in such a way as to reproduce the perturbative expansion of a time-space noncommutative theory. This involves imaginary dilatons and inverted space-time light-cone coordinates, not so dissimilar from the inversion occurring near the horizon of a black hole. A dual supergravity master field in the appropriate large- $N$  strong-coupling limit is obtained. It has a naked singularity, whose possible resolution by stringy effects might reinstate the ‘custodial effect’.

## 2 Two Families of Critical Field Singularities

The classical open-string dynamics in a background with metric  $g_{\mu\nu}$  and NS  $B_{\mu\nu}$ -fields is controlled by the open-string metric  $G_{\mu\nu}$  and the noncommutativity parameter  $\theta^{\mu\nu}$ , which determine the world-sheet propagator. At the tree level:

$$\langle X^\mu(\tau) X^\nu(\tau') \rangle_{\text{disk}} = -\alpha' G^{\mu\nu} \log(\tau - \tau')^2 + \frac{i}{2} \theta^{\mu\nu} \epsilon(\tau - \tau'). \quad (2.3)$$

The open-string parameters are related to the sigma-model fields  $g_{\mu\nu}, B_{\mu\nu}$  by

$$G^{\mu\nu} + \frac{\theta^{\mu\nu}}{2\pi\alpha'} = \left( \frac{1}{g + 2\pi\alpha' B} \right)^{\mu\nu}, \quad (2.4)$$

with  $\theta^{\mu\nu}$  entering the commutation relations of the string zero modes as in (1.1). Therefore, the noncommutativity properties of the open-string theory are associated to non-vanishing NS  $B$ -field backgrounds. In particular, in order to induce time-space noncommutativity  $\theta^{0i} \neq 0$ , we need non-vanishing electric components of the NS  $B$ -field.

For magnetic  $B$ -field backgrounds,  $B_{0i} = 0 = \theta^{0j}$ , one can expose the noncommutative properties by explicitly disentangling the  $\theta$ -nonlocality from the stringy fuzziness. Such a decoupling was studied in full detail by Seiberg and Witten in [2]. It involves a zero-slope limit of the string theory  $\alpha' \rightarrow 0$ , holding fixed the open-string metric  $G_{\mu\nu}$  and the open-string NC parameter  $\theta^{\mu\nu}$ , resulting in a low-energy noncommutative Yang–Mills theory, with interactions specified in terms of Moyal products. In this limit

$$\theta^{\mu\nu} \rightarrow \left( \frac{1}{B} \right)^{\mu\nu}. \quad (2.5)$$

It is also found necessary to scale the string coupling so that

$$G_s = g_s \left( \frac{-\det(g + 2\pi\alpha' B)}{-\det(g)} \right)^{\frac{1}{2}} \quad (2.6)$$

is the effective string-loop expansion parameter in the limit. The zero-slope limit involves scaling the sigma-model metric to zero as  $g_{\mu\nu} \sim (\alpha')^2$  and leaving  $B_{\mu\nu}$  fixed. Therefore, there seems to be a clear obstruction when the  $B$ -field has electric components. In that case, the matrix

$$g + b \equiv g + 2\pi\alpha' B$$

will eventually have imaginary eigenvalues, and the open-string background is ill-defined. To be more specific, suppose that  $B_{\mu\nu}$  is skew-diagonalized and consider the time-like  $2 \times 2$  block. Parametrizing this block by

$$(g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu}) = \begin{pmatrix} -\bar{g} & -\bar{b} \\ \bar{b} & \bar{g} \end{pmatrix}, \quad (G^{\mu\nu}) \equiv \begin{pmatrix} -\bar{G}^{-1} & 0 \\ 0 & \bar{G}^{-1} \end{pmatrix}, \quad (\theta^{\mu\nu}) \equiv \begin{pmatrix} 0 & \bar{\theta} \\ -\bar{\theta} & 0 \end{pmatrix}, \quad (2.7)$$

one finds the relations

$$\bar{G}^{-1} = \frac{\bar{g}}{\bar{g}^2 - \bar{b}^2}, \quad \bar{\theta} = -2\pi\alpha' \frac{\bar{b}}{\bar{g}^2 - \bar{b}^2} \quad (2.8)$$

and

$$G_s = g_s \sqrt{\frac{\bar{g}^2 - \bar{b}^2}{\bar{g}^2}}. \quad (2.9)$$

Therefore, any zero-slope limit in which the  $B$ -field dominates over the metric,  $g + b \approx b$ , leads to a positive determinant  $\det(g + b) > 0$  and to an imaginary effective string coupling (2.9). In particular, at the vanishing locus of the determinant,  $\det(g + b) = 0$ , the open-string inverse metric and noncommutative parameters are infinite!

We find that the moduli space is divided into two branches: one with  $\det(g + b) < 0$ , continuously connected to the fully commutative background, and one with  $\det(g + b) > 0$ , which seems to be ill-defined. The critical line is characterized by a singularity of the open-string parameters  $G, \theta$ .

Open strings in electric-field backgrounds are well-known to exhibit a classical singularity at a critical value of the electric field [7, 12]. This singularity can be spotted in the low-energy effective dynamics as the vanishing of the Dirac–Born–Infeld (DBI) Lagrangian

$$\mathcal{L}_{\text{DBI}} \sim \frac{1}{g_s} \sqrt{-\det(g + b)} = 0 \quad (2.10)$$

or, in T-dual language, as the limiting value of the speed of light for the corresponding T-dual D-brane [6]. Thus, the critical line of singularities coincides with the classical singularity of the DBI action. The physical interpretation is that the D-brane becomes effectively tensionless, and in fact tachyonic for  $\det(g + b) > 0$ .

Actually, the DBI singularity at the locus  $\det(g + b) = 0$  does not exhaust all the singularities in the mapping (2.8), for it is not well-suited for studying the limits in the  $(g, b)$  moduli space where both  $g$  and  $b$  diverge. One can factor the volume form and study the submanifold

$$\det(1 + g^{-1} b) = 0, \quad (2.11)$$

which includes extra singularities at  $g \sim \infty$ . In fact, there is a whole moduli space of them. Solving for the inverse relations in (2.8):

$$\bar{g} = \frac{\bar{G}}{1 - \left(\frac{\bar{G}\bar{\theta}}{2\pi\alpha'}\right)^2}, \quad \bar{b} = -\frac{\bar{G} \left(\frac{\bar{G}\bar{\theta}}{2\pi\alpha'}\right)}{1 - \left(\frac{\bar{G}\bar{\theta}}{2\pi\alpha'}\right)^2}, \quad (2.12)$$

we find that there is a family of singularities at  $\bar{G}\bar{\theta} = 2\pi\alpha'$ , which are solutions of (2.11) and correspond to divergent sigma-model backgrounds. We shall denote these singularities as ‘sigma-model singularities’,  $g$ -singularities for short, on account of the fact that  $g, b$  diverge at finite values of the open-string parameters  $G, \theta$ . The previously identified singularities at finite  $g, b$  will be referred to as ‘open-string singularities’, or  $G$ -singularities for short, since there it is  $G^{-1}$  and  $\theta$  that diverge.

The effective loop-expansion parameter  $G_s$  vanishes at both types of critical points. For the  $G$ -singularity this is obvious from the definition in (2.9). For the  $g$ -singular points we find

$$G_s = g_s \sqrt{1 - \left(\frac{\bar{G}\bar{\theta}}{2\pi\alpha'}\right)^2}. \quad (2.13)$$

Therefore, a classical theory (no string-loop corrections) is expected at the singularities, unless  $g_s$  is scaled so as to compensate for the vanishing of the determinant factors.

In spite of this, the perturbative physics (at fixed and small string coupling  $g_s \ll 1$ ) near these two families of singularities is rather different. First, we have noticed that the  $G$ -singularities are characterized by the vanishing of the effective D-brane tension. On the other hand, at the  $g$ -singularities, we have the opposite behaviour, a divergent D-brane tension:

$$\sqrt{-\det(g + 2\pi\alpha' B)} = \sqrt{\bar{g}^2 - \bar{b}^2} = \sqrt{\frac{\bar{G}^2}{1 - \left(\frac{\bar{G}\bar{\theta}}{2\pi\alpha'}\right)^2}} = \sqrt{\frac{-\det(G)}{1 - \left(\frac{\bar{G}\bar{\theta}}{2\pi\alpha'}\right)^2}}. \quad (2.14)$$

This is consistent with the idea that  $g$ -singularities are less harmful than  $G$ -singularities, at least when considering open-string dynamics. Open-string perturbation theory is a weak-field expansion in the background of a D-brane with some fluxes. If the effective tension of the brane vanishes, the fluctuations are too violent and nonlinear effects cannot be controlled in perturbation theory. We shall confirm this picture below when we study the scaling of the general perturbative amplitude at a  $G$ -singular point.

The differences also show up at the level of the free spectrum. This can be studied by direct quantization of the free open string in background fields [11, 13, 7]. A shortcut to the answer can be obtained from a reinterpretation of the one-loop vacuum amplitude, *i.e.* the annulus [13]:

$$\mathcal{A} = Z[g = \eta, b = 0] \sqrt{-\det(g)} \cdot \left(\frac{-\det(g + b)}{-\det(g)}\right) \quad (2.15)$$

with  $\eta$  the Minkowski metric, and

$$Z[g = \eta, b = 0] = \frac{1}{2} \int_0^\infty \frac{dt}{t} \text{Tr } e^{-t\alpha'(p_\mu \eta^{\mu\nu} p_\nu + M^2)} \quad (2.16)$$

the vacuum Minkowski amplitude. In terms of the open-string metric,

$$G = (g + b) \frac{1}{g} (g + b), \quad (2.17)$$

we can use the identities

$$\frac{\det(G)}{\det(g + b)} = \frac{\det(g + b)}{\det(g)} = \left(\frac{\det(G)}{\det(g)}\right)^{\frac{1}{2}} \quad (2.18)$$

to rewrite the complete expression in the form of the modified partition function:

$$\mathcal{A} = \frac{1}{2} \int_0^\infty \frac{dt}{t} \text{Tr } e^{-t\alpha'(p_\mu G^{\mu\nu} p_\nu + M^2)}. \quad (2.19)$$

Thus, the effect of the background fields at the level of free-string propagation amounts simply to modifying the dispersion relation to

$$G^{\mu\nu} p_\mu p_\nu + M^2 = 0. \quad (2.20)$$

Now, since this open-string metric  $G_{\mu\nu}$  is singular at the  $G$ -singularity, we have a degenerate dispersion relation there, whereas the free propagation of open strings in the  $G$ -space-time is completely smooth at the  $g$ -singularities.

One can generalize the comparison between the two types of singularities to higher-loop corrections. The general perturbative amplitude with  $L$  open-string loops comes from a spherical worldsheet with  $L+1$  holes and  $n$  open-string vertex insertions. We can determine the rough features of the dependence on the background fields by a scaling argument.

First, there is a factor of

$$\sqrt{-\det(g)}$$

from the translational zero modes; a factor of

$$\left( \frac{-\det(g+b)}{-\det(g)} \right)^{\frac{1}{2}}$$

for each boundary, from the normalization of boundary states, as in ref. [13] (see also [14]). A factor of  $g_s^{L-1}$  from the dilaton; and a factor of  $\lambda^n$  from the normalization of the vertex operators, to be determined later on.

All together:

$$\mathcal{A}_{L,n} \sim \sqrt{-\det(g)} \left( \frac{-\det(g+b)}{-\det(g)} \right)^{\frac{L+1}{2}} \cdot g_s^{L-1} \cdot \lambda^n. \quad (2.21)$$

The normalization of the vertex operators depends on what we consider as a background. We have seen that, in the vicinity of the  $G$ -singularity, the open-string metric  $G_{\mu\nu}$  is singular and therefore the effective Lagrangian is not naturally constructed as a density to be integrated against  $\sqrt{-\det(G)}$ . Instead, we must use the original sigma-model metric and write the effective action as

$$S_{\text{eff}} = \int dx \sqrt{-\det(g)} \mathcal{L}_{\text{eff}}. \quad (2.22)$$

From the general expression (2.21) we see that all vacuum amplitudes, including the vacuum disk that determines the bare D-brane tension, vanish at the  $G$ -singularity. Regarding interactions, the canonical wave-function normalization of open-string fluctuations should be fixed by requiring that the tree-level two-point function density contains only the volume factor, *i.e.* we require

$$\mathcal{A}_{0,2} \sim \sqrt{-\det(g)} \left( \frac{-\det(g+b)}{-\det(g)} \right)^{\frac{1}{2}} g_s^{-1} \lambda^2 \sim \sqrt{-\det(g)},$$

which determines

$$\lambda \sim (g_s)^{\frac{1}{2}} \cdot \left( \frac{-\det(g+b)}{-\det(g)} \right)^{-\frac{1}{4}}. \quad (2.23)$$

Thus, in the vicinity of a  $G$ -singular point, amplitudes that are ‘sufficiently quantum’, *i.e.* at high loop order ( $L > -1+n/2$ ), will vanish, while those that are ‘sufficiently nonlinear’, *i.e.* with a large number of external legs ( $n > 2L+2$ ), will diverge. Therefore, we confirm our expectations based on the vanishing of the D-brane effective tension at the  $G$ -singularities: the purely classical nonlinear effects blow up.

Incidentally, it is interesting to notice that a further scaling of the string coupling  $g_s \rightarrow 0$  can stabilize the effective expansion parameter for the nonlinearities  $\lambda$ . In that limit the theory becomes completely classical, albeit with complicated nonlinear interactions, and it becomes also decoupled from closed strings.

On the other hand, at a  $g$ -singular point, it is the open-string metric that is the smooth one, and we are led to writing an effective Lagrangian as a density with respect to the  $G_{\mu\nu}$  metric. Using the identities (2.18) in (2.21) we find

$$\mathcal{A}_{L,n} \sim \sqrt{-\det(G)} G_s^{L-1} \lambda^n. \quad (2.24)$$

The wave-function normalization of vertex operators is now determined with respect to the  $\sqrt{-\det(G)}$  volume element:

$$\mathcal{A}_{0,2} \sim \sqrt{-\det(G)} G_s^{-1} \lambda^2 \sim \sqrt{-\det(G)},$$

which yields the expected

$$\lambda \sim (G_s)^{\frac{1}{2}}, \quad (2.25)$$

*i.e.* in this case the same effective coupling governs both the nonlinearities and the quantum corrections. Since  $G_{\mu\nu}$  is regular, the amplitudes at the  $g$ -singular point scale as

$$\mathcal{A}_{L,n} \rightarrow (G_s)^{L-1+n/2}. \quad (2.26)$$

This vanishes if  $L - 1 + n/2 > 0$ . The vacuum disk (brane tension) diverges, as we have pointed out before. From the point of view of the theory on the brane, this is just an infinite adjustment of the vacuum energy and, unlike the phenomenon of vanishing tension at the  $G$ -singular points, it should be harmless. Also, the vacuum annulus amplitude is finite, as it corresponds to the smooth propagation of open strings in the  $G$ -metric, according to the dispersion relation (2.20). All interactions and loop corrections vanish.

Therefore, the theory at the  $g$ -singular point is free. Again, we can scale an interacting theory by further tuning  $g_s$ . In this case we must take the underlying string-theory background to strong coupling.

In summary, we have seen that the set of critical electric field singularities has two branches, lying on the boundary of each other’s moduli space. The perturbative physics is qualitatively different at these two branches. The most spectacular difference is the fact that the brane tension diverges at  $g$ -singularities, with smooth open-string propagation, whereas the brane becomes tensionless at the  $G$ -singularities, where open-string dynamics completely degenerates.

### 3 Noncommutative Censorship by Stringy Fuzziness

We are chiefly interested in investigating under which conditions the noncommutativity scale can be decoupled from the string scale, *i.e.* whether one can take a limit in which  $|\theta| \gg \alpha'$ , and

whether this limit is described by a noncommutative field theory equipped with Moyal products in the time direction.

Naively, we would like to emulate the prescription of ref. [2] and, starting from a smooth sigma-model specified by a non-singular pair  $(g, b) = (g, 2\pi\alpha' B)$  and a real  $G_s$ , we would take the limit of vanishing metric at fixed  $B$ . In doing so, we have found that one hits the wall of  $G$ -singularities, specified by the locus of  $\det(g + b) = 0$ . In the vicinity of this point we indeed have  $|\theta| \gg \alpha'$ , but the open-string metric  $G_{\mu\nu}$  degenerates, so that the low-energy description in terms of Moyal products does not exist along the lines of [2].

In fact, if we stay clear from the  $G$ -singular points, *i.e.* if we require  $G_{\mu\nu}$  to be fixed and smooth, then we still hit the  $g$ -singularities. From the inverse relations (2.12) we see that, starting at sub-stringy noncommutativity  $|\theta| \ll \alpha'$ , any attempt at increasing it at fixed  $G_{\mu\nu}$  takes us into a  $g$ -singular point. At this point the noncommutativity is not decoupled from the stringy fuzziness. Rather, it is of the same order:

$$|\theta|_{\max} = 2\pi\alpha', \quad (3.27)$$

where we have normalized the fixed open-string metric to the Minkowski metric value. Beyond this point, one can make  $|\theta|$  larger than  $\alpha'$ , but the effective coupling  $G_s$  becomes imaginary and one does not expect to find a unitary theory.

Therefore, we see a kind of ‘censorship’: string theory works very hard in order to keep the length scale of time-space noncommutativity hidden below the normal stringy fuzziness at the string length.

## 4 A Formal Decoupling Limit

Quantum field theories with time-space noncommutativity have no obvious Hamiltonian formulation and, therefore, unitarity is a nontrivial consistency issue for these theories. Since one finds difficulties in embedding *decoupled* time-space noncommutative field theories into a well-defined string background, one is led to suspect that these theories are truly inconsistent at the quantum level.

Still, the perturbation theory is obtained from that of purely spatial noncommutativity in what may be a deceptively simple manner: we just continue a spatial coordinate to  $x^j \rightarrow ix^0$ , and at the same time also take  $\theta^{jk} \rightarrow i\theta^{0k}$ , in order to have a real  $\theta^{\mu\nu}$  matrix after the rotation. In this process the real value of the Yang–Mills coupling does not change.

The same manipulation takes the open-string perturbation theory, when written in terms of  $G_{\mu\nu}, \theta^{\mu\nu}$  and  $G_s$ , from purely spatial to time-space noncommutativity. Namely one performs an analytic continuation at fixed  $G_s$ . It is then natural to ask what this operation entails for the underlying string sigma-model, *i.e.* the ‘closed-string’ parameters  $g_{\mu\nu}, B_{\mu\nu}, g_s$ . From the general relations (2.12) and (2.13), we see that the formal zero-slope limit  $\alpha' \rightarrow 0$  with constant  $G, \theta, G_s$  is

$$\bar{g} \rightarrow -(2\pi\alpha')^2 \frac{\bar{G}}{\theta^2}, \quad \bar{B} \rightarrow \frac{1}{\theta}, \quad G_s \rightarrow \sqrt{-g_s^2 \left( \frac{\bar{G}\bar{\theta}}{2\pi\alpha'} \right)^2}. \quad (4.28)$$

In particular, the equation for the sigma-model metric components  $\bar{g}$  forces them to have opposite sign to those of the open-string metric  $\bar{G}$ . In addition, a real Yang–Mills coupling or, equivalently, a real  $G_s$ , requires an analytic continuation to imaginary values of the ‘closed-string’ coupling  $g_s$ .

Therefore, the formal scaling limit that induces the time-space noncommutative Moyal products is the Seiberg–Witten scaling limit, supplemented with two rather unconventional operations:

- An analytic continuation of the closed-string coupling into imaginary values  $g_s^2 \rightarrow -g_s^2$ .
- A switch of space-time signature, or ‘tumbling’ of light cones in the time-space noncommutative plane.

These two features are strongly reminiscent of black-hole horizons. Namely, the ‘tumbling’ of light cones is a standard feature of horizons, and the continuation to imaginary  $g_s$  is related to the tachyonic character of the D-brane in this region. It is tempting to interpret the  $(g_{\mu\nu})$ -frame and the  $(G_{\mu\nu})$ -frame as living in opposite sides of a horizon, so that a tachyonic particle (as seen in the  $(g_{\mu\nu})$ -frame) can escape through the horizon and emerge on the other side as a standard non-tachyonic particle with respect to the  $(G_{\mu\nu})$ -frame.

Although complex values of the dilaton seem like a rather exotic feature, it is not completely obvious that the combined action of the two above operations would yield an inconsistent theory, when reinterpreted in the  $G_{\mu\nu}$  space-time. Equivalently, it is not obvious that the open-string perturbation theory with time-space noncommutative phases and real  $G_s$  would be inconsistent.

At the level of the low-energy effective theory, this is a question of whether the perturbation theory with time-space noncommutative Moyal products defines a consistent  $S$ -matrix. Apparently acausal effects were reported in [8] at the tree level, but similar effects show up in the Veneziano amplitude for open strings. Therefore, these effects are not necessarily fatal to the quantum  $S$ -matrix. The most likely candidate for a smooth theory would be  $\mathcal{N} = 4$  super-Yang–Mills, which is largely safe from the infrared singularities of [15], owing to the improved ultraviolet behaviour. For the spatially noncommutative  $U(N)$  theory we also have a candidate large- $N$  master field via the AdS/CFT correspondence [16, 17]. At large values of the ’t Hooft coupling  $\lambda_{\text{YM}} = g_{\text{YM}}^2 N$ , it is given by the near-horizon D3-brane backgrounds in type IIB string theory with  $B$ -fields [19, 18].

We can construct an AdS/CFT dual of the time-space noncommutative theory starting from the supergravity solution of  $N$  D3-branes with electric NS  $B$ -field backgrounds [19]. In the string frame:

$$ds^2 = \frac{1}{\sqrt{H(r)}} \left( d\mathbf{y}^2 + f(r)(-dt^2 + dx^2) \right) + \sqrt{H(r)} \left( dr^2 + r^2 d\Omega_5^2 \right) \quad (4.29)$$

with

$$H(r) = 1 + \left( \frac{R}{r} \right)^4, \quad \frac{1}{f(r)} = -\frac{\text{sh}^2(\alpha)}{H(r)} + \text{ch}^2(\alpha). \quad (4.30)$$

The  $B$ -field and dilaton profiles are

$$\bar{b} = 2\pi\alpha' \bar{B} = -\text{th}(\alpha) \frac{f(r)}{H(r)}, \quad e^{2\phi} = g_s^2 f(r). \quad (4.31)$$

The charge radius satisfies  $R^4 = 4\pi g_s N(\alpha')^2$ .

The zero-slope limit in eq. (4.28) can be implemented by rescaling the coordinates as

$$(-dt^2 + dx^2) \rightarrow -\left(\frac{2\pi\alpha'}{\theta}\right)^2 (-dt^2 + dx^2).$$

This induces a corresponding rescaling of  $\bar{B}$  by a factor of  $-(2\pi\alpha'/\bar{\theta})^2$  (from the tensor transformation), whereas the analytic continuation in  $g_s$  induces

$$e^{2\phi} \rightarrow -e^{2\phi}.$$

In order to satisfy the  $B$ -field boundary condition in (4.28) we must take  $\text{th}(\alpha) \rightarrow \infty$ . This can be achieved by defining  $t_\theta$  such that

$$t_\theta \equiv \text{th}(\alpha), \quad \text{sh}^2(\alpha) = \frac{t_\theta^2}{1-t_\theta^2}, \quad \text{ch}^2(\alpha) = \frac{1}{1-t_\theta^2}.$$

and continuing  $t_\theta$  outside the interval  $[-1, 1]$  as

$$t_\theta \rightarrow \frac{\bar{\theta}}{2\pi\alpha'}.$$

The complete scaling limit is  $\alpha' \rightarrow 0$  with the previous prescriptions, plus the standard AdS/CFT scaling  $H \rightarrow 1/(Ru)^4$ , with  $u$  a sliding energy scale of the strongly coupled theory.

The scaling limit defines the strong-coupling noncommutativity length  $a_\theta$ :

$$(a_\theta)^4 = R^4 t_\theta^2 \rightarrow \lambda_{\text{YM}} \left( \frac{\bar{\theta}}{2\pi} \right)^2, \quad (4.32)$$

with  $\lambda_{\text{YM}} = g_{\text{YM}}^2 N = 4\pi G_s N$  the 't Hooft coupling of the field theory.

The final result is the dual background:

$$\frac{ds^2}{R^2} = u^2 \left( d\mathbf{y}^2 + \hat{f}(u)(-dt^2 + dx^2) \right) + \frac{du^2}{u^2} + d\Omega_5^2 \quad (4.33)$$

$$\bar{B} = \frac{1}{\bar{\theta}} (a_\theta u)^4 \hat{f}(u), \quad e^{2\phi} = \left( \frac{\lambda_{\text{YM}}}{4\pi N} \right)^2 \hat{f}(u) \quad (4.34)$$

where the nontrivial profile function is

$$\hat{f}(u) = \frac{1}{1 - (a_\theta u)^4}. \quad (4.35)$$

We see that this is just the analytic continuation of the purely spatial noncommutative master-field metric of refs. [19, 18] under

$$y \rightarrow it, \quad \theta^{yx} \rightarrow i\theta^{tx} = i\bar{\theta}, \quad B_{yx} \rightarrow -iB_{tx}. \quad (4.36)$$

This is natural, given the fact that the supergravity background is the effective description induced by the sum over planar diagrams. Since the analytic continuations (4.36) generate, diagram by diagram, the time-space noncommutative perturbation theory, it is not surprising that the resulting master field is obtained by the same analytic continuation. Turning the argument around, we can say that this serves as a nontrivial consistency check of the zero-slope formulas in (4.28).

The large- $N$  master field encoded in eqs. (4.33-35) has two important physical properties: First, like the magnetic- $B$  counterpart, the geometry approaches the standard  $AdS_5 \times S^5$  in the

infrared  $u \rightarrow 0$ . This suggests that indeed this theory shows no dangerous infrared singularities of the type discussed in [15].

The second important property is the existence of a naked singularity at  $u = a_\theta^{-1}$ , which is absent in the case of the magnetic counterpart. The supergravity approximation breaks down badly at physical length scales of the order of the strong-coupling noncommutativity scale  $a_\theta$ , with the metric  $B$ -field and dilaton blowing up. In fact, the entire ‘ultraviolet’ region at  $u > a_\theta^{-1}$  is hard to interpret, since the local value of the closed-string coupling  $e^{\phi(u)}$  becomes imaginary. Notice also the ‘tumbling’ of light cones in this region, in much the same fashion as the prescription for the microscopic parameters in (4.28).

Thus, the microscopic  $g$ -singularity seems to reappear, even after the zero-slope limit (4.28) has been taken, disguised as a threshold singularity in the low-energy large- $N$  master field. Therefore, there are grounds to suspect that field theories with time-space noncommutativity may be ultraviolet-inconsistent at the quantum mechanical level, at least within the approximations involved in the above derivation (large- $N$  and large ‘t Hooft coupling).

It is interesting to consider the behaviour of thermodynamic quantities in the vicinity of the naked singularity. The near-extremal version of (4.33) can be obtained by the substitution  $dt^2 \rightarrow h(u)dt^2$ ,  $du^2 \rightarrow du^2/h(u)$ , with  $h(u) = 1 - (u_0/u)^4$  a Schwarzschild-like profile function. As the horizon at  $u = u_0$  approaches the singularity, the thermodynamic functions scale with the parameter  $\xi_0 = 1 - (a_\theta u_0)^4$ . In particular, the entropy and specific heat *vanish* as  $S \sim C_V \sim (V/a_\theta^3) \sqrt{\xi_0}$  while the internal energy diverges logarithmically  $E \sim -(V/a_\theta^4) \log(\xi_0)$ . On the other hand, the Hawking temperature diverges when the horizon reaches the singularity:  $T \sim 1/(a_\theta \sqrt{\xi_0})$ . Therefore, as long as  $T \lesssim a_\theta^{-1}$ , the large- $N$  Euclidean path integral with fixed-temperature boundary conditions is not dominated by the singular geometry. This would suggest that the theory makes sense if defined with a cutoff  $u < u_\Lambda < a_\theta^{-1}$  that stays well below the noncommutativity energy scale. On the other hand, the fact that the large- $N$  extensive thermodynamic functions depend on  $\theta$  (through  $a_\theta$ ) contradicts the perturbative argument of [20], namely Moyal phases should cancel in planar vacuum diagrams. The same is true of the formally continued  $a_\theta \rightarrow ia_\theta$  metrics that, though nonsingular, show other nonstandard thermodynamical features, such as a maximum temperature and a branch of negative specific heat (see refs. [19, 21]). It is tempting to interpret these pathologies as further evidence for the quantum inconsistency of the time-space noncommutative field theories. In this case, the lack of a Hamiltonian formalism would be responsible for the failure of the thermodynamical interpretation of the periodically identified Euclidean backgrounds.

The naked singularity does not protect itself within the supergravity approximation. For example, one can calculate the static action of a D3 probe brane in this background,

$$S_{\text{probe}} = T_{\text{D3}} \int e^{-\phi} \sqrt{-\det(g + 2\pi\alpha' B)} + T_{\text{D3}} \int C_4^{RR} + T_{\text{D3}} \int C_2^{RR} \wedge 2\pi\alpha' B, \quad (4.37)$$

and find a flat potential as a function of the radial coordinate  $u$ . This means that the singularity is reachable by Higgs expectation values in the breaking of the  $U(N)$  group into  $U(1)$  factors.

We see two possible resolutions of this situation. Either the singularity is a true one and signals an inconsistency of the time-space noncommutative field theory, or the full type IIB closed-string theory resolves the singularity. In that case we are presumably back into our ‘censorship’ criterion: stringy fuzziness is fully apparent at the time-space noncommutativity scale.

**Note added:** While this paper was being prepared for publication, there appeared two papers [22] that had substantial overlap with our results. In particular, these articles propose a scaling of an *interacting* theory at the  $g$ -singularities, still with a time-space noncommutativity parameter of the order of the string scale, as in (3.27). In an S-dual description, this is related to a purely space-space noncommutative Yang–Mills theory [22, 23]. The large- $N$  master fields turn out to be the Lorentzian time-space noncommutative metrics in [19]. Since thermodynamic functions are invariant under duality transformations, we see that the thermodynamics of the scaled theory at the  $g$ -singularity seems to be independent of  $\theta$  to the leading order in the large- $N$  limit.

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## References

- [1] A. Connes, *Noncommutative Geometry*, Academic Press (1994).
- [2] N. Seiberg and E. Witten, *JHEP* **9909** (1999) 032, [hep-th/9908142](#).
- [3] A. Connes, M.R. Douglas and A. Schwarz, *JHEP* **9802** (1998) 003, [hep-th/9711162](#).
- [4] M.R. Douglas and C. Hull, *JHEP* **9802** (1998) 008, [hep-th/9711165](#).
- [5] M. Li, [hep-th/9802052](#). Y.-K. E. Cheung and M. Krogh, *Nucl. Phys.* **B528** (1998) 185, [hep-th/9803031](#). F. Ardalan, H. Arfaei and M.M. Sheikh-Jabbari, [hep-th/9803067](#); *JHEP* **9902** (1999) 016, [hep-th/9810072](#); [hep-th/9906161](#). C.-S. Chu and P.-M. Ho, *Nucl. Phys.* **B550** (1999) 151, [hep-th/9812219](#); *Nucl. Phys.* **B568** (2000) 447, [hep-th/9906192](#). V. Schomerus, *JHEP* **9906** (1999) 030, [hep-th/9903205](#).
- [6] C. Bachas, *Phys. Lett.* **B374** (1996) 37, [hep-th/9511043](#).
- [7] C.P. Burgess, *Nucl. Phys.* **B294** (1987) 427. V.V. Nesterenko, *Int. J. Mod. Phys.* **A4** (1989) 2627. C. Bachas and M. Petrati, *Phys. Lett.* **B296** (1992) 77.
- [8] N. Seiberg, L. Susskind and N. Toumbas, [hep-th/0005015](#).
- [9] T. Yoneya, [hep-th/0004074](#).
- [10] C.-S. Chu and P.-M. Ho, *Nucl. Phys.* **B550** (1999) 151, [hep-th/9812219](#). M.M. Sheikh-Jabbari, *Phys. Lett.* **B455** (1999) 129, [hep-th/9901080](#). D. Bigatti and L. Susskind, [hep-th/9908056](#). Z. Yin, *Phys. Lett.* **B466** (1999) 234, [hep-th/9908152](#).
- [11] E.S. Fradkin and A.A. Tseytlin, *Phys. Lett.* **B163** (1985) 123. A. Abouelsaood, C.G. Callan, C.R. Nappi and S.A. Yost, *Nucl. Phys.* **B280** (1987) 599.
- [12] S. Gukov, I.R. Klebanov and A.M. Polyakov, *Phys. Lett.* **B423** (1998) 64, [hep-th/9711112](#).
- [13] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, *Nucl. Phys.* **B288** (1987) 525.
- [14] O. Andreev, [hep-th/0001118](#).
- [15] S. Minwalla, M. Van Raamsdonk and N. Seiberg, [hep-th/9912072](#).
- [16] J.M. Maldacena, *Adv. Theor. Math. Phys.* **2** (1998) 231, [hep-th/9711200](#).
- [17] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Phys. Lett.* **B428** (1998) 105, [hep-th/9802109](#). E. Witten, *Adv. Theor. Math. Phys.* **2** (1998) 253, [hep-th/9802150](#).
- [18] A. Hashimoto and N. Itzhaki, [hep-th/9907166](#).

- [19] J.M. Maldacena and J.G. Russo, *JHEP* **9909** (1999) 025, [hep-th/9908134](#).
- [20] D. Bigatti and L. Susskind, [hep-th/9908056](#).
- [21] J.L.F. Barbón and E. Rabinovici, *JHEP* **04** (1999) 015, [hep-th/9910019](#).
- [22] N. Seiberg, L. Susskind and N. Toumbas, [hep-th/0005040](#). R. Gopakumar, J. Maldacena, S. Minwalla and A. Strominger, [hep-th/0005048](#).
- [23] O.G. Ganor, G. Rajesh and S. Sethi, [hep-th/0005046](#).